A Generalization of the Kepler Problem

Guowu Meng

Department of Mathematics, Hong Kong Univ. of Sci. and Tech.

Clear Water Bay, Kowloon, Hong Kong

Email: mameng@ust.hk

March 14, 2008

Abstract

We construct and analyze a generalization of the Kepler problem. These generalized Kepler problems are parameterized by a triple (D, κ, μ) where the dimension $D \geq 3$ is an integer, the curvature κ is a real number, the magnetic charge μ is a half integer if D is odd and is 0 or $1/2$ if D is even. The key to construct these generalized Kepler problems is the observation that the Young powers of the fundamental spinors on a punctured space with cylindrical metric are the right analogues of the Dirac monopoles.

1 Introduction

The Kepler problem, owing to its significant role in the major developments of physics in the last three centuries, is a well-known scientific problem. It has been known for quite a while that the Kepler problem and its beautiful mathematical structure are not isolated: there are deformations/generalizations along several

directions. Firstly, there is a generalization along the curvature direction [1, 2], i.e., deforming the configuration space from the Euclidean space to the non-Euclidean space of constant curvature κ; secondly, there is a generalization along the dimension direction [3], i.e., deforming the dimension (of the configuration space) from three to a generic integer $D \geq 3$; thirdly, there is a generalization along the magnetic charge direction [4, 5], i.e., deforming the magnetic charge (of the nucleus of a hypothetic hydrogen atom) from 0 to a generic half integer μ.

The Kepler problem has been generalized to the following cases in the literature:

1. Case $\mu = 0$, $\kappa \geq 0$, generic D, see Refs. [6, 7],
2. Case $D = 3$, $\kappa \geq 0$, generic μ, see Ref. [8],
3. Case $D = 3$, $\kappa < 0$, generic μ, see Ref. [9] (where dimension five case was also indicated),
4. Case $D = 5$, $\kappa = 0$, generic μ, see Ref. [10].

Note that, if $\mu \neq 0$, the generalized Kepler problems are not expected to exist in dimensions other than three and five. That is because, to get the Kepler problem with magnetic charge in dimension other than three, the only method one used so far is to apply the symmetry reduction to the isotropic harmonic oscillator in a higher dimension. For example, the isotropic harmonic oscillator in dimension eight admits a natural SU(2)-symmetry whose symmetry reduction to

$$\mathbb{R}^8 \setminus \{\vec{0}\}/\mathrm{SU}(2) = \mathbb{R}^5 \setminus \{\vec{0}\}$$

yields the five dimensional Kepler problem with magnetic charge [10]. Now the group SU(2) is the space of unit quaternions with its action on $\mathbb{R}^8$ $(= \mathbb{H}^2)$ being

the scalar multiplication, and the set of quaternions, $\mathbb{H}$, is an associative division algebra over $\mathbb{R}$. Since there are only two nontrivial associative division algebras over $\mathbb{R}$ (the complex numbers and the quaternions), the symmetry reduction method can only produce the Kepler problems with nonzero magnetic charges in dimensions three and five, which correspond to the complex numbers and the quaternions respectively.

The main purpose here is to show that the Kepler problem and its mathematical structure have a generalization beyond what is currently known in the literature. More explicitly, we claim that there is a generalized Kepler problem for each triple (D, μ, κ), where $D \geq 3$ is an integer, κ is a real number, and μ is a half integer if D is odd and is 0 or 1/2 if D is even; moreover, these generalized Kepler problems all share the characteristic mathematical beauty of the original Kepler problem.

The key to our generalization in this generality lies in our recent observation [11] that the Young powers[1] of the fundamental spinors on a punctured space with cylindrical metric are the correct analogues[2] of the Dirac monopoles, see also Ref. [12]. With this observation in mind, it is basically clear that how one can construct our generalized Kepler problems. The details are given in the next section.

Our generalized Kepler problems are solved here by the factorization method of Schrödinger [1, 13]. In principle, they can be solved by other methods such as the algebraic method of Pauli [14] and the path integral method [15, 16], though

[1]Given an irreducible representation V of the orthogonal group, the k-th Young power of V is the irreducible representation whose highest weight vector is equal to k multiple of the highest weight vector of V.

[2]It is interesting to note that, the magnetic charge, while it could be any half integer in odd dimensions, it must be 0 or 1/2 in even dimensions.

these other methods are technically more difficult. Since the main purpose of this article is to inform the community of the existence of the generalized Kepler problems in this full generality, here we are contented with showing only the factorization method in details and leave the solution by other methods for the future.

2 Generalized Kepler problems

From the physics point of view, a generalized Kepler problem is obtained from the Kepler problem by adding suitable background gravity field plus magnetic field and making a suitable adjustment to the scalar Coulomb potential so that the extra large hidden symmetry is still preserved. The background gravity field is introduced via the Riemannian metric for the sphere or pseudo-sphere; and the background magnetic field is just the spin connection of the cylindrical metric on the configuration space that we have mentioned in the introduction. The configuration space is the punctured Euclidean space if $\kappa = 0$ or the punctured pseudo-sphere if $\kappa < 0$ or the sphere with two poles removed; i.e., topologically it is either a punctured coordinate space if $\kappa \geq 0$ or a punctured (open) disk if $\kappa < 0$. For uniform treatment, we shall project the configuration space isometrically onto the coordinate space. Note that, the cylindrical metric is introduced *only* for the purpose of introducing the high dimensional analogue of Dirac's magnetic monopoles; we could have introduced the background magnetic field by just writing down the mysterious and hard-to-remember explicit formulae without mentioning this true mathematical content. With this in mind, we are now ready to give the detailed presentation of the generalized Kepler problem.

Let κ be a real number. The *configuration space* is a D-dimensional Riemannian manifold $(Q_\kappa(D), ds_\kappa^2)$ with constant curvature κ, here $Q_\kappa(D)$ is $\mathbb{R}^D - \{0\}$

if $\kappa \geq 0$ or $\{x \in \mathbb{R}^D \mid 0 < |x| < \frac{1}{\sqrt{-\bar\kappa}}\}$ if $\kappa < 0$, and

$$ds_\kappa^2 = \frac{dx_0^2 + \cdots + dx_{D-1}^2}{(1+\bar\kappa r^2)^2}, \tag{1}$$

where $r = |x|$ and $\bar\kappa = \frac{\kappa}{4}$.

Let $ds^2 = \frac{dx_0^2+\cdots+dx_{D-1}^2}{r^2}$. Just as in Ref. [17], we observe that $(Q_\kappa(D), ds^2)$ is the product of a line interval with the unit round sphere S^{D-1}. When D is odd, we let $\mathcal{S}_\pm$ be the positive/negative spinor bundle of $(Q_\kappa(D), ds^2)$, and when D is even, we let $\mathcal{S}$ be the spinor bundle of $(Q_\kappa(D), ds^2)$. As we noted in Ref. [17] that these spinor bundles come with a natural $\mathrm{SO}(D)$ invariant connection — the Levi-Civita spin connection of $(Q_\kappa(D), ds^2)$. As a result, the Young product of I copies of these bundles, denoted by $\mathcal{S}_+^I$, $\mathcal{S}_-^I$ (when D is odd) and $\mathcal{S}^I$ (when D is even) respectively, also comes with a natural connection. For more details, the readers may consult Ref. [17].

For the sake of notational sanity, from here on, when D is odd and μ is a half integer, we rewrite $\mathcal{S}_+^{2\mu}$ as $\mathcal{S}^{2\mu}$ if $\mu \geq 0$ and rewrite $\mathcal{S}_-^{-2\mu}$ as $\mathcal{S}^{2\mu}$ if $\mu \leq 0$; moreover, we adopt this convention for $\mu = 0$: $\mathcal{S}^0$ is the product complex line bundle with the product connection.

Introducing the pre-potential

$$V_\kappa = -\frac{1}{r} + \bar\kappa r \tag{2}$$

and factor

$$\delta_\mu = \begin{cases} (n-1)|\mu| + \mu^2 & \text{when } D = 2n+1 \\ (n-1)\mu & \text{when } D = 2n\,, \end{cases} \tag{3}$$

we are then ready to make the following definition.

Definition 1. *Let $D \geq 3$ be an integer, κ be a real number, μ be a half integer if D is odd and be 0 or $1/2$ if D is even. The* **generalized Kepler problem**

labeled by (D, κ, μ) is defined to be a quantum mechanical system on $Q_\kappa(D)$ for which the wave-functions are sections of $\mathcal{S}^{2\mu}$, and the hamiltonian is

$$\hat{h} = -\frac{1}{2}\Delta_\kappa + \frac{1}{2}\delta_\mu \left(V_\kappa^2 + \kappa\right) + V_\kappa \,, \tag{4}$$

where Δ_κ is the negative-definite Laplace operator on $(Q_\kappa(D), ds_\kappa^2)$ twisted by $\mathcal{S}^{2\mu}$.

Remark that this hamiltonian is an essentially self-adjoint operator when some suitable "boundary conditions" are imposed on the wave functions, see Ref. [18] for the case $\mu = 0$ and Ref. [19] for the general case. However, we are not concerned about this issue here because we can solve the eigenvalue problem explicitly.

3 Spectral analysis

We use the analytic method pioneered by Schrödinger [1] together with the textbook knowledge of representation theory for compact Lie groups to obtain the spectra for the bound states of our generalized Kepler problems. As in Ref. [17], we assume that the small Greek letters, α, β, etc., run from 0 to $D-1$ and the small Latin letters, a, b, etc., run from 1 to $D-1$. Under a local gauge on $Q_\kappa(D)$ minus the negative 0-th axis, the gauge potential[3], $A = A_\alpha dx_\alpha$, is explicitly calculated in Ref. [17] as follows:

$$A_0 = 0, \qquad A_b = -\frac{1}{r(r+x_0)} x_a \gamma_{ab} \,, \tag{5}$$

where $\gamma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$, and $\gamma_a = i\vec{e}_a$ with $\vec{e}_a$ being the element in the Clifford algebra that corresponds to the a-th standard coordinate vector of $\mathbb{R}^{D-1}$.

Let $\nabla_\alpha = \partial_\alpha + iA_\alpha$. Then we can write down the explicit local formula for

[3]We use the Einstein convention: the repeated index is always summed over.

Δ_κ as follows:

$$\Delta_\kappa = (1+\bar{\kappa}r^2)^D \nabla_\alpha \frac{1}{(1+\bar{\kappa}r^2)^{D-2}} \nabla_\alpha. \tag{6}$$

In view of that fact that $x_\alpha A_\alpha = 0$, when written in terms of the polar coordinates, the hamiltonian $\hat{h}$ then becomes

$$-\frac{(1+\bar{\kappa}r^2)^D}{2r^{D-1}}\partial_r(1+\bar{\kappa}r^2)^{2-D}r^{D-1}\partial_r + \frac{(1+\bar{\kappa}r^2)^2}{2r^2}(\frac{1}{2}\hat{L}^2_{\alpha\beta} - \bar{c}_2 + \delta_\mu) + V_\kappa,$$

where

$$\hat{L}_{\alpha\beta} = -i(x_\alpha\nabla_\beta - x_\beta\nabla_\alpha) + r^2 F_{\alpha\beta} \tag{7}$$

and $\bar{c}_2$ is the value of the quadratic Casimir operator of so$(D-1)$ on the $2|\mu|$-th Young power of the fundamental spin representation of so$(D-1)$.

Let $L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{D-1}})$ be the L^2-sections of vector bundle $\mathcal{S}^{2\mu}$ restricted to the unit sphere S^{D-1}. From representation theory (for example, §129 of Ref. [20]), we know that

$$L^2(\mathcal{S}^{2\mu}|_{\mathrm{S}^{D-1}}) = \bigoplus_{l\geq 0} \mathscr{R}_l\,, \tag{8}$$

where, when D is odd, $\mathscr{R}_l$ is the irreducible representation space of Spin(D) having the highest weight equal to $(l+|\mu|, |\mu|, \cdots, |\mu|)$; and when D is even, $\mathscr{R}_l = \mathscr{R}_l^+ \oplus \mathscr{R}_l^-$ with $\mathscr{R}_l^\pm$ being the irreducible representation space of Spin(D) having the highest weight equal to $(l+\mu, \mu, \cdots, \pm\mu)$. It is then clear that the Hilbert spaces $\mathscr{H}$ of the bound states is a subspace of

$$\bigoplus_{l\geq 0} \mathcal{H}_l \tag{9}$$

with $\mathcal{H}_l$ being a subspace of $L^2(\mathrm{I}_\kappa, d\mu) \otimes \mathscr{R}_l$. Here I_κ is the half line of the positive real numbers if $\kappa \geq 0$ and is the open interval $(0, \frac{1}{\sqrt{-\bar{\kappa}}})$ if $\kappa < 0$, and

$$d\mu = (\frac{r}{1+\bar{\kappa}r^2})^D \frac{dr}{r}. \tag{10}$$

On $\mathcal{H}_l$, $\frac{1}{2}L^2_{\alpha\beta}$ is a constant; in fact, from the computation done in Ref. [17], we have

$$c := \frac{1}{2}L^2_{\alpha\beta} - \bar{c}_2 + \delta_\mu = m(m+1) - \frac{(D-1)(D-3)}{4} \tag{11}$$

with $m = l + |\mu| + \frac{D-3}{2}$; so the Schrödinger equation becomes an ODE for the radial part:

$$\left(-\frac{(1+\bar{\kappa}r^2)^D}{2r^{D-1}}\partial_r \frac{r^{n-1}}{(1+\bar{\kappa}r^2)^{D-2}}\partial_r + \frac{(1+\bar{\kappa}r^2)c}{2r^2} + V_\kappa - E_{kl}\right) R_{kl} = 0\,, \tag{12}$$

where E_{kl} is the k-th eigenvalue for a fixed l and the additional label $k \geq 1$ is introduced for the purpose of listing the radial eigenfunctions, just as in the Kepler problem. The further analysis follows the factorization method in Ref. [13]. Making the transformation

$$y = (\frac{r}{1+\bar{\kappa}r^2})^{\frac{D-1}{2}} R_{kl}, \qquad dx = \frac{1}{1+\bar{\kappa}r^2}\, dr, \qquad \lambda = 2E_{kl}, \tag{13}$$

the preceding equation becomes

$$\frac{d^2y}{dx^2} + r(x,m)y + \lambda y = 0 \tag{14}$$

with

$$r(x,m) = \frac{2(1-\bar{\kappa}r^2)}{r} - \frac{(1+\bar{\kappa}r^2)^2 c + (\frac{D-1}{2})^2(1-\bar{\kappa}r^2)^2 - \frac{D-1}{2}(1+\bar{\kappa}r^2)^2}{r^2}. \tag{15}$$

By substituting the value for c given in equation (11), we have

$$\begin{aligned}
\frac{r(x,m)}{-\kappa} &= -\tfrac{m(m+1)}{\sinh^2\sqrt{-\kappa}x} + \tfrac{1}{\sqrt{-\bar{\kappa}}}\coth\sqrt{-\kappa}x - (\tfrac{D-1}{2})^2 && \text{for } \kappa < 0,\\
\frac{r(x,m)}{\kappa} &= -\tfrac{m(m+1)}{\sin^2\sqrt{\kappa}x} + \tfrac{1}{\sqrt{\bar{\kappa}}}\cot\sqrt{\kappa}x + (\tfrac{D-1}{2})^2 && \text{for } \kappa > 0,\\
r(x,m) &= \tfrac{2}{r} - \tfrac{m(m+1)}{r^2} && \text{for } \kappa = 0.
\end{aligned} \tag{16}$$

Let $I_0(\kappa)$ be ∞ if $\kappa \geq 0$ or be the integer part of $\left((-\kappa)^{-\frac{1}{4}} - |\mu| - \frac{D-1}{2}\right)$ if $\kappa < 0$. Using the factorization method [13] plus theorems 2 and 3 in §129 of

Ref. [20], one can arrive at the following theorem for the generalized Kepler problems.

Theorem 1. *For the generalized Kepler problem labeled by* (D, κ, μ)*, assume* $I_0(\kappa) \geq 0$*, then the following statements are true:*

1) The discrete energy spectra are

$$E_I = -\frac{1}{2(I + \frac{D-1}{2} + |\mu|)^2} + \frac{(I + \frac{D-1}{2} + |\mu|)^2 - (\frac{D-1}{2})^2}{2}\kappa\,, \tag{17}$$

where I *is a non-negative integer and is less than or equal to* $I_0(\kappa)$*;*

2) The Hilbert space $\mathscr{H}$ *of the bound states admits a linear* $\mathrm{Spin}(D+1)$*-action under which there is a decomposition*[4]

$$\mathscr{H} = \bigoplus_{I=0}^{I_0(\kappa)} \mathscr{H}_I$$

where $\mathscr{H}_I$ *is a model for the irreducible* $\mathrm{Spin}(D+1)$*-representation whose highest weight is* $(I + |\mu|, |\mu|, \cdots, |\mu|, \mu)$*;*

3) The linear action in part 2) extends the manifest linear action of $\mathrm{Spin}(D)$*, and the finite dimensional complex vector space* $\mathscr{H}_I$ *in part 2) is the eigenspace of the hamiltonian* $\hat{h}$ *with eigenvalue* E_I *in equation (17).*

Remark that there are continuous spectra unless $\kappa > 0$.

Acknowledgment

I would like to thank Armen Nersessian for being interested in my work on Kepler problems. I would also like to thank the conference organizer for inviting me to write this report.

[4]This follows from a fact in representation theory: the direct sum of certain suitable irreducible representation spaces of $\mathrm{Spin}(D)$ forms an irreducible representation for $\mathrm{Spin}(D+1)$. In other word, we get the extra large symmetry here from the representation theory without working out the symmetry generators explicitly.

References

[1] E. Schrödinger, *Pro. R. Irish Acad. A* **46** (1940), 9.

[2] L. Infeld and A. Schild, *Phys. Rev.* **67** (1945), 121.

[3] E.C.G. Sudarshan, N. Mukunda and L. O'Raifeartaigh, *Phys. Lett.* **19** (1965), 322.

[4] D. Zwanziger, Exactly soluble nonrelativistic model of particles with both electric and magnetic charges, *Phys. Rev.* **176** (1968), 1480-1488.

[5] H. McIntosh and A. Cisneros, Degeneracy in the presence of a magnetic monopole, *J. Math. Phys.* **11** (1970), 896-916.

[6] P. W. Higgs, *J. Phys. A: Math. Gen.* **12** (1979), 309-323.

[7] H. I. Leemon, *J. Phys. A: Math. Gen.* **12** (1979), 489-501.

[8] V. V. Gristsev, Yu. A. Kurochkin, V. S. Otchik, *J. Phys. A: Math. Gen.* **33** (2000), 4903-4910.

[9] A. Nersessian and G. Pogosyan, On the relation of the oscillator and Coulomb systems on (pseudo) spheres, *Phys. Rev. A* **63**, 020103(R) (2001). *E-print*, quant-ph/0006118.

[10] T. Iwai, The geometry of the SU(2) Kepler problem, *J. Geom. Phys.* **7** (1990), 507-535.

[11] G. W. Meng, Dirac and Yang monopoles revisited. *CEJP* **5** (2007), pp. 570-575. *E-print*, math-ph/0409051.

[12] I. I. Cotaescu: Generalized Dirac monopoles in non-Abelian Kaluza-Klein theories. *Nucl. Phys. B* **719** (2005) 140-164.

[13] L. Infeld and T. E. Hull, *Rev. Mod. Phys.* **23** (1951), 21-68.

[14] W. Pauli, *Z. Phys.* **24** (1926), 336.

[15] C. Grosche, F. Steiner, *Handbook of Feynman Path Integrals*, Springer Tracts in Modern Physics, **145** (1998)

[16] A.O. Barut, A. Inomata and G. Junker, *J. Phys. A: Math. Gen.* **20** (1987), 6271.

[17] G. W. Meng, The MICZ-Kepler problems in all dimensions. *J. Math. Phys.* **48** (2007), 032105. *E-print*, math-ph/0507028, version 3. A minor corection about the allowable vaules of μ in even dimensions has been given in version 4 of this E-print.

[18] T. Kato, Fundamental properties of Hamiltonian operators of Schrödinger type. *Trans. Amer. Math. Soc.* **70** (1951), 195 -211.

[19] M. Braverman, O. Milatovich and M. Shubin, Essential selfadjointness of Schrödinger-type operators on manifolds. *Russian Math. Surveys* **57** (2002), 641-692.

[20] D. P. Zelobenko, *Compact Lie groups and their representations*, Translations of mathematical monographs v. **40**, American Mathematical Society, 1973.